Learning Pathways in Numeracy:

Addressing Early Numeracy Skills

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Why we created this document. "Numeracy" is a term that refers to all the mathematics that young students learn including number, operations, and geometry and measurement concepts. This *Learning Pathways in Numeracy* document was created by the Office of Superintendent of Public Instruction (OSPI) primarily as a tool to help teachers and parents to understand the role of progressions in developing numeracy skills in children. Having knowledge of how children progress in their early development of numeracy concepts helps teachers and parents to select and use activities that will **intentionally** build numeracy skills in children. Building these skills is foundational for children as they progress through their study of mathematics.

This document is designed to facilitate discussions related to progression pathways and to indicate the body of concepts that can be developed in young children. A study within a professional learning community of the resource documents (see page 10) used to create these pathways would be a recommended way to deepen understanding of each pathway. An informed teacher could then use this document as a quick reference to isolate children's learning along a particular pathway. Determining the next steps in learning and the associated instructional tasks to accomplish that learning would move children forward along a particular pathway.

Why mastery of math concepts is crucial at an early age. Research has shown that children are capable of learning math concepts at a much earlier age than has been previously recognized. A study also has indicated that when controlling for IQ, family income, gender, temperament, type of previous educational experience, and whether children came from single or two parent families, **the mastery of early math concepts upon school entry was the strongest predictor of future academic success**.[1] In 2012, Washington State began collecting observational data through the Washington Kindergarten Inventory of Developing Skills (WaKIDS) around early math concepts on large cohorts of kindergarten students within the first months of school entry. The results showed that many Washington State children are entering kindergarten with limited numeracy skills. With this knowledge, there was a growing need for OSPI to develop ways to help teachers and parents improve the numeracy skills of pre-school children.

How this document is connected to the Common Core. The *Learning Pathways in Numeracy* was written using the Common Core State Standards (CCSS) domain titles (Counting and Cardinality, Operations and Algebraic Thinking, etc.). This was done to show that foundational concepts that support Common Core kindergarten standards and beyond can be learned at an early age. While the *Learning Pathways for Numeracy* gives some key benchmarks for a given age-range, there are intermediary stages between these benchmarks. These intermediary stages are treated in greater depth in the resources that were used to develop the pathways (see page 10). In addition, the pathways for kindergarten through third grade in this document follow directly from the CCSS. The CCSS were built from progressions of learning that can be found at http://ime.math.arizona.edu/progressions/. OSPI considers these progressions required reading for teachers at each grade level.

It should be noted that while the *Learning Pathways in Numeracy* is divided into mathematical domains, there is much cross-over between these domains. This linear model does not accurately represent those connections or the coherence found between domains. Reading the listed resources and the CCSS progressions will help teachers make those connections.

Developmental progressions are a guide, not an absolute. Although the resources used for this document agree that there is a progression of learning for early numeracy concepts, the period when a particular concept can be developed may be slightly different from resource to resource. It needs to be understood that the *Learning Pathways for Numeracy* is not meant to be definitive about the age in which these concepts can be developed and so there can be discrepancies between documents. The general progression should be what is considered and not necessarily the age given. Children develop their cognitive knowledge at different rates and the suggested age ranges are only that.

Our hope is that this *Pathways* document will be the start of a discussion and then a movement to help children develop their interest and capacity to learn meaningful mathematics in their early years.

[1] School readiness and later achievement.
Duncan, Greg J.; Dowsett, Chantelle J.; Claessens, Amy; Magnuson, Katherine; Huston, Aletha C.; Klebanov, Pamela; Pagani, Linda S.; Feinstein, Leon; Engel, Mimi; Brooks-Gunn, Jeanne; Sexton, Holly; Duckworth, Kathryn; Japel, Crista
Developmental Psychology, Vol 43(6), Nov 2007, 1428-1446. http://dx.doi.org/10.1037/0012-1649.43.6.1428

2014. Developed by Julie Wagner in coordination with the Mathematics Learning and Teaching Department at the Office of Superintendent of Public Instruction (OSPI). OSPI would like to thank Douglas Clements, Ph.D., University of Denver, for reviewing a draft version of this document

Learning Pathways for Counting and Cardinality

Counting and Cardinality

	Counting	ELG	GOLD™	Subitizing/ Early Operations	ELG	GOLD™	Comparing and Ordering	ELG	GOLD™
0 – 36 months	• Imitates rote counting using some names of numbers. • Verbally counts to 5, may be incorrect beyond this. • Verbally counts to 10. • Keeps 1-1 correspondence for 5 or less objects in a line. Beyond this, may counts quickly at the end if knows more numbers than objects, or recycles words if number of objects is greater than numbers known by rote. ₃	9–18 m 16–36 m 16–36 m	20a OY 20a YG 20a YG	• Demonstrates understanding of the concepts of one, two, and more. • Recognizes and names the number of items in a set of two or three. • Adds and subtracts very small collections (up to 3) nonverbally. ₃	16–36 m	20b OY 20b GB 20b GB	• Understands the idea of "more" related to food or play. • Puts objects in 1-to-1 or 1-to-many in provoked correspondence. ₃ • Knows more/less for very small collections of items, or with big differences in number of items. • Uses comparison words • Compares collections of 1 – 4 items if the collections are made up of the same objects. ₃	9–18 m 16–36 m	20b OY 22 OYG
3 – 4 years	• Counts verbally to 10 or beyond. • Understands cardinality in counting, for at least 5 objects. • Counts out 5 objects.	3–4 yr 3–4 yr	20a YG 20b GB 20a YG	• Instantly recognizes and names the number of items in a set of three or four • Makes a small collection with the same number as another collection	3–4 yr	20b GB	• Uses gestures or words to make comparisons • Compares groups of 1 -5 by matching or counting when objects in each group are about the same size. • Accurately counts two equal collections, but when asked, says the collection of larger objects has more. ₃	3–4 yr	22GB
4 – 5 years	• Counts verbally to 20 and beyond. • Counts 1020 objects accurately. • Gives next number in sequence (1 - 10). • Counts out 10 objects. • Identifies numerals 110. • Writes some numerals and connects each to counted objects.	4–5 yr 4–5 yr 4–5 yr 4–5 yr	20a BP 20a BP 20a BP 20c BP 20cBP	• Instantly recognizes and names the number of items in a set of five. • Makes a collection with the same number as another collection. • Quickly names parts of any whole up to 5, or the whole given the parts. ₃	4–5 yr	20b GB	• Uses comparative language (more, less, same) to compare collections up to 10 by counting, even when the collection with the larger quantity of objects is made up of smaller objects. • Orders three to six objects by one attribute.	4–5 yr 4–5 yr	20b BP 22 GB
	• Counts verbally to 100 by ones and tens. * • Counts forward from a given number.* • Writes and represents numbers to 20 and beyond.* • Counts to tell "How many?" to at least 20. * • Counts out objects to 20. *	5–K 5–K 5–K	20aP 20aP 20cP 20aP						

Counting and Cardinality

	Counting	ELG	GOLD™	Subitizing/ Early Operations	ELG	GOLD™	Comparing and Ordering	ELG	GOLD™
Grade 1	• Counts to 120 starting from any number. * Reads and writes numerals to 120.* • Represents a number of objects with a written numeral.*	1st G 1st G		• Uses conceptual subitizing to compose and decompose numbers, and to understand place value and operations.			• Determines "how many more" or "how many less" by addition or subtraction in comparing situations.		

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Learning Pathways for Number and Base Ten

Number and Operations in Base Ten/Fractions

	Place Value	ELG	GOLD™	Place Value and Operations	ELG	GOLD™	Fractions	ELG	GOLD™
Kindergarten	• Composes and decomposes numbers 11-19 into tens and ones.*								
Grade 1	• Understands that 2-digit numbers represent amounts of tens and ones. * • Compares two two-digit numbers based on meanings of tens and ones using >, <, and =.*	1st G		• Adds within 100 including adding a 2-digit number and a 1-digit number, and adding a 2-digit number and a multiple of 10.* • Finds 10 more or less than a number, explaining reasoning.*	1st G		• Partitions circles and rectangles into two and four equal shares describing the whole and parts by number of shares.* • Understands that decomposing into more shares creates smaller shares.* (Found in Geometry domain)	1st G	
Grade 2	• Understands place value for 3-digit numbers.* • Counts within 1000; skip counts by 5s, 10s, and 100s.* • Reads and writes numbers to 1000.* • Compares two three-digit numbers based on meanings of hundreds, tens, and ones digits using <, >, and =.*	2nd G		• Fluently adds and subtracts within 100 using strategies.* • Using concrete models, adds or subtracts within 1000.* • Mentally adds or subtracts 10 or 100 from any given number 100- 900.* • Explains why addition and subtraction strategies work.*	2nd G 2nd G 2nd G		• Recognizes that equal shares of identical wholes need not have the same shape.* • Partitions circles and rectangles into two, three, or four equal shares and names them.* (Found in Geometry domain)		
Grade 3	• Uses place value to round whole numbers to nearer 10 or 100.*			• Fluently adds or subtracts within 1000.* • Multiplies 1-digit numbers by multiples of 10 from 10-90.*	3rd G		• Develops understanding of fractions as numbers on a number line.* • Understands unit fractions.* • Understands, recognizes, and generates simple equivalent fractions.* • Compares two fractions with the same numerator or denominator.* • Recognizes that comparisons are valid only when referring to the same whole.* • Partitions shapes into parts with equal area and expresses each part as a unit fraction.*	3rd G	

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Learning Pathways for Operations and Algebraic Thinking

Operations and Algebraic Thinking						
	Operations	ELG	GOLD™	Algebraic Thinking	ELG	GOLD™
0 – 36 months	• Uses gestures to ask for more. • Understands getting more or having less.	9-18m	20b O			
3 – 4 years	• Knows that a whole is bigger than its parts. • Finds the total sum of small groups of items. • Combines and separates up to five objects and describes the parts.	3-4 yr 3-4 yr	20b GB	• Intuitively has understanding that when adding two numbers, it does not matter which number "comes first." ₃		
4 – 5 years	• Finds results for joining and take-away by using concrete objects and counting all.	4-5 yr	20b GB	• Intuitively has understanding that when adding three numbers it does not matter which two you add first. ₃		
5-Kindergarten	• Uses counting-on and counting-up-to strategies to find results. • Fluently adds and subtracts within 5.* • Represents addition and subtraction within 10 with objects.* • Makes sets of 6 – 10 objects and describes the parts, identifying which part has more, less, or the same.	5 – K 5 - K	20b BP 20b BP	• Makes algebraic generalizations (such as, subtracting zero from any number gives that number, or subtracting a number from itself gives zero), with guidance. ₃		
Grade 1	• Represents and solves all addition and subtraction situations (result unknown, change unknown, start unknown) within 20.* • Adds and subtracts within 20 using a variety of informal and intuitive strategies and can describe the strategies used.* • Demonstrates fluency within 10.* • Works with addition and subtraction equations with the unknown in all positions.*	1st G 1st G	20b P	• Understands and applies properties of operations and the relationship between addition and subtraction.* • With guidance, makes algebraic generalizations, such as any number added to and then subtracted from x, leaves x. ₃		
Grade 2	• Solves one- and two-step word problems within 100 with unknowns in any situation and subtype.* • Describes thinking when solving a word problem. • Adds and subtracts within 20, knowing facts by end of year.*	2nd G 2nd G 2nd G		• Works with equal groups of objects to gain foundations for multiplication.*		
Grade 3	• Represents and solves problems involving multiplication and division.* • Multiplies and divides within 100.* • Multiplies a single digit by a multiple of 10.* • Solves problems involving the four operations.*	3rd G 3rd G 3rd G 3rd G		• Understands properties of multiplication and the relationship between multiplication and division.* • Identifies and explains patterns in arithmetic.*		

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Learning Pathways for Measurement and Data (Measurement)

Measurement and Data						
	Measurement	ELG	GOLD™	Geometric Measurement	ELG	GOLD™
0 – 36 months	• Attends to overall appearance of size, labeling as big/little. • Explores measuring tools, such as measuring cups, or a ruler.	9-18m	22 OYG	• Manipulates shapes individually, but does not combine to compose a larger shape.		
3 – 4 years	• Compares size by sight, feel, and comparing to hands, feet, etc. • Compares and orders a small set of objects as appropriate according to size, length, weight, area, volume. • Knows usual sequence of basic daily events. • Knows a few ordinal numbers.	3-4 yr 3-4 yr	22 GB 22 GB 22 GB	• Fills simple pattern block puzzles using trial and error.[3]		
4 – 5 years	• Physically aligns two objects to determine which is longer or if same length. [3] • May be able to measure with ruler, but often lacks understanding or skill. [3]	4-5 yr	22 BP	• Puts several shapes together to make one part of a picture. [3]		
5 and Kinder-garten	• Describes measureable attributes of objects.* • Directly compares two objects.* • Uses ordinal numbers from *first* to *tenth.*[3] • Uses measurement words and some standard measurement tools accurately.		22 P 22 P	• Analyzes and compares two-dimensional and three-dimensional shapes in different sizes and orientations.* • Models shapes by building them from component parts.* • Composes simple shapes to make larger shapes.*		21b BP 21b P 21b P
Grade 1	• Measures lengths indirectly and by iterating length units without gaps or overlaps. * • Tells and writes time in hours and half-hours.*	1st G		• Composes two-dimensional and three-dimensional shapes to create a composite shape.*	1st G	
Grade 2	• Measures and estimates lengths in standard units.* • Relates addition and subtraction to length and the number line.* • Tells time to the nearest 5 minutes.* • Solves problems involving money.* • Recognizes that comparisons are valid only when referring to the same unit.*	2nd G 2nd G		• Partitions a rectangle into rows and columns of same size squares and count to find total.*		
Grade 3	• Tells and writes time to the nearest minute and measures time intervals.* • Solves addition and subtraction time interval problems on a number line.* • Measures and estimates liquid volumes and masses of objects using grams, kilograms, and liters.* • Solves problems of all types that involve measurement.*			• Understands concepts of area and relates area to multiplication and to addition.* • Recognizes perimeter as an attribute of plane figures and solves area and perimeter problems.*	3rd G 3rd G	

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Learning Pathways for Measurement and Data (Data)

Measurement and Data						
	Categorical Data	ELG	GOLD™	Measurement Data	ELG	GOLD™
3 – 4 years	• Sorts and/or describes objects by a non-geometric attribute (size, color) or by shape.	3-4 yr	13 GB			
4 – 5 years	• Sorts objects by one attribute.	4-5 yr	13 GB			
5 and Kinder-garten	• Sorts and classifies objects into more than one category and counts the number in each.* • Resorts objects into new categories. ₃	5 - K	13 BP			
Grade 1	• Organizes, represents and interprets data with up to three categories.*		13 P			
Grade 2	• Draws a bar and/or picture graph and answers questions related to them.*			• Generates measurement data by measuring length to the nearest whole unit.* • Shows length measurements on a line plot marked with whole number units, connecting this to the representation of whole numbers on a number line.*		
Grade 3	• Draws scaled picture and bar graphs to represent a data set and solves problems.*			• Generates data by measuring lengths marked with whole numbers, halves and fourths and shows on a line plot, connecting this to the representations of fractions on a number line.*		

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Learning Pathways for Geometry

Geometry						
	Shapes	ELG	GOLD™	Spatial Relationships/Structuring	ELG	GOLD™
0 - 36 months	• Puts things together, such as simple matching puzzles, nesting cups. • Matches familiar shapes (circle, square, typical triangle) with the same size and orientation. • Recognizes and names circles and squares, maybe triangles. • Matches shapes by rotating to prototype. • Judges shapes the same if they are more visually similar than different. ₃	9-18m 16-36m 16-36m 16-36m	13 OY 21b OY 21b G	• Follows simple directions related to position *(in, on, up, down*).	9-18m 16-36m	20a OY
3 – 4 years	• Recognizes more shapes in real-world, less typical triangle shapes, and some rectangles with same size and orientation. • Recognizes some shapes with different sizes and orientation. ₃	3-4 yr	21b BP	• Follows simple directions related to proximity (*behind, under, beside, next to, between*).	3-4 yr	21a YG
4 – 5 years	• Recognizes and compares most familiar shapes and typical examples of other shapes, such as cubes and sphere. • Composes simple shapes to form larger shapes. • Names at least some three-dimensional shapes. • Recognizes sides and angles as distinct geometric objects.₃		21b BP 21b BP	• Identifies positions of objects in space by using words like, *beside, inside, next to, above, below, under.* • Uses and responds to positional words.	4-5 yr	21a BP
5 and Kindergarten	• Names shapes regardless of orientation.* • Identifies shapes as two-dimensional or three-dimensional.* • Recognizes shapes in the environment.	5 - K 5 - K	21b BP 21b BP 21b BP	• Uses and makes simples sketches to locate objects. • Correctly uses position words to describe objects.	5 - K	21a P
Grade 1	• Distinguishes between defining and non-defining attributes.*					
Grade 2	• Recognizes and draws shapes having specified attributes.*			• Mentally constructs an array that is tiled with squares that line up in row and columns. ₃		
Grade 3	• Understands that shapes in different categories may share attributes.* • Recognizes rhombuses, rectangles, and squares as quadrilaterals.*			• Distinguishes between linear and area measures.		

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Learning Pathways for Mathematical Practices

	Ways to incorporate in the classroom:	Questions that could be asked:
1. Make sense of problems and persevere in solving them.	• Give students problems worthy of being solved. • Let students develop their own plan to solve a problem.	• What do you notice about….? • Tell me what you have already done. What would be a good next step? • Is there another way to do this?
2. Reason abstractly and quantitatively.	• Expect students to model, interpret, and connect multiple representations.	• How did you decide what to do? • What do the numbers (or objects or pictures) represent? • Explain your model to me.
3. Construct viable arguments and critique the reasoning of others.	• Create a safe environment so that all students feel free to explain their thinking and to critique the reasoning of others. • Have students use objects or pictures to justify reasoning. • Ask students to justify their own reasoning.	• How do you know? • What is the same? • What is different? • Will it still work if…?
4. Model with mathematics.	• Expect students to be able to model a situation. • Have a number of students show and explain their models.	• How can you represent this? • Could you use these (manipulatives, blocks, counters) to represent this situation?
5. Use appropriate tools strategically.	• Use technology and math tools (manipulatives, measurement tools, calculators) when appropriate.	• What is your estimate? • Is there any tool that might help you solve this?
6. Attend to precision.	• Use precise mathematical language in the classroom and expect the same of students. • Ask students to identify units used.	• Is your answer reasonable? • How do you know? • Explain how your solution answers the question.
7. Look for and make use of structure.	• Encourage students to find numerical patterns in their work. • Ask students to look for the component parts to a problem and explain each.	• What do you notice? • Do you see any patterns? • Where have you seen this before? • What do you already know that would help you solve this?
8. Look for an express regularity in repeated reasoning.	• Encourage students to generalize numerical patterns. • Ask students to predict answers.	• Is this always true? • Can you find a rule for this? • What would you predict to be the solution? Why?

Resources

Adapted from:
1) *Washington State Early Learning and Development Guidelines: Birth through 3rd Grade 2012, Washington State Department of Early Learning,* Olympia, WA – denoted as ELG.
2) *Teaching Strategies GOLD® Objectives for Development & Learning: Birth Through Kindergarten.* (2010). Teaching Strategies, LLC, Bethesda, MD–denoted as GOLD ™. Used with permission of Teaching Strategies, LLC. Any use without prior written approval is strictly prohibited.
3) *Learning and Teaching Early Math: The Learning Trajectories Approach* (2nd ed.)(2014), Douglas Clements and Julie Sarama, Routledge, New York, NY – denoted with a subscript 3.
4) *Early Childhood Mathematics Education Research: Learning Trajectories for Young Children* (2009), Julie Sarama and Douglas Clements, Routledge, New York, NY.
5) *Mathematics Learning in Early Childhood: Paths toward Excellence and Equity* (2009), National Research Council, The National Academies Press, Washington, DC.
6) *First Steps in Mathematics: Measurement* (2007), Department of Education and Training of Western Australia.
7) *Curriculum Focal Points for Prekindergarten* (2010), NCTM, Reston, VA.
8) *Common Core State Standards for Mathematics*, Grades K-3, (2009), Chief State School Officers and National Governors Association Center for Best Practices - denoted with asterisks.
9) *Implementing Standards for Mathematical Practices*, Institute for Advanced Study/ Park City Mathematics Institute/Created by Learning Services, Modified by Melisa Hancock 2013.
10) Common Core progression documents, http://ime.math.arizona.edu/progressions/.